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Gauge Mechanical Model in N Dimensions

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The Christ–Lee mechanical model is generalized to N spatial dimensions. Its quantization as a gauge system is carried out, emphasizing the relationship between gauge-fixing and curvilinear coordinates in configuration space.

1. INTRODUCTION

It is in the context of field theories that the concept of local symmetries or gauge invariances first arose. Through the past decades, it has acquired the status of a central principle. However, gauge invariance is not restricted to this kind of theory and, in particular, it may be found in elementary classical and quantum mechanics (Dirac, 1964). Since field theories are both conceptually and technically the most subtle branches of contemporary physics, simple models are introduced as a guide to our intuition and formal manipulations.

In this paper, we treat a mechanical model invariant under time-dependent rotations in real N-dimensional space, i.e., an $SO(N)$-gauge invariant system. This model was previously introduced in two dimensions and in spite of its simplicity it may be used as a first example to test different approaches to quantization (Alessandrini, 1989; Bouzas, 1990; Lee, 1981; Prokhorov, 1982). We extend it to any number of spatial dimensions as an illustration to the treatment of various features of gauge systems. We show how it is quantized and how it can be interpreted in terms of classical configuration space. In this respect, the relation between local symmetry and curvilinear coordinate systems and its consequences for quantization are emphasized.

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In Section 2 we present the model and discuss it classically. The meaning of gauge invariance and gauge conditions is analyzed there. In Section 3 the quantization is performed in the Schrödinger picture, one that is not frequent in infinite-dimensional problems. The discussion of quantization and its connection with gauge invariance extends to Section 4. To make the treatment self-contained, in the Appendices we work out some mathematical details needed in the main text.

2. THE MODEL AND ITS CLASSICAL TREATMENT

The system we shall be considering (Lee, 1981; Prokhorov, 1982) consists of a single point particle with mass $m=1$ moving in N-dimensional real space under the influence of a central force derived from a potential function V. We start from its Lagrangian

$$L=\tfrac{1}{2}\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}-V(|\mathbf{x}|) \tag{1}$$

where $\mathbf{x}=(x_1,\ldots,x_N)\in\mathbb{R}^N$ is the position vector of the particle, $\dot{\mathbf{x}}$ stands for $d\mathbf{x}/dt$, and the dot indicates the usual inner product in $\mathbb{R}^N$. This Lagrangian is invariant under N-dimensional rotations, i.e., under transformations of the form

$$\mathbf{x}'=U\mathbf{x} \tag{2}$$

with $U\in SO(N)$, the group of orthogonal $N\times N$ matrices with determinant $+1$. The algebra associated to this group is the algebra of antisymmetric $N\times N$ matrices Y. Any such matrix Y may be uniquely written in terms of a basis $\{T^a\}$ $(a=1,\ldots,\binom{N}{2})$ of the algebra as

$$Y=y_aT^a; \qquad y^a\in\mathbb{R} \tag{3}$$

where the summation convention over repeated indices is understood. The basis $\{T^a\}$ may be chosen such that the antisymmetric matrices T^a satisfy the normalization condition

$$\operatorname{tr}(T^a\cdot T^b)=-\delta^{ab} \tag{4}$$

We then define the Lagrangian of our model to be (Prokhorov, 1982)

$$L=\tfrac{1}{2}(\mathscr{D}\mathbf{x})\cdot(\mathscr{D}\mathbf{x})-V(|\mathbf{x}|) \tag{5}$$

with

$$\mathscr{D}\mathbf{x}\equiv\left(\frac{d}{dt}-Y\right)\cdot\mathbf{x}=\dot{\mathbf{x}}-T\cdot\mathbf{x} \tag{6}$$

The components y^a ot Y and those of $\mathbf{x}$ are the degrees of freedom of this system. $\mathscr{D}\mathbf{x}$ is the "covariant derivative" of $\mathbf{x}$.

The matrix U in (2) that leaves the Lagrangian (1) invariant depends on $\binom{N}{2}$ angles α_a parametrizing the group $SO(N)$. In Appendix C a particular parametrization is given. If we let the α's depend on time, U defines a transformation to a rotating coordinate system and the Lagrangian (1) is no longer invariant under (2). This is what is meant by gauge invariance, namely, invariance under a group of transformations whose parameters are arbitrary functions of time.

The Lagrangian (5), on the other hand, is unchanged by (2) provided we specify that the new degrees of freedom y_a transform according to

$$\begin{aligned} \mathbf{x}' &= U \cdot \mathbf{x} \\ Y' &= U \cdot Y \cdot {}^tU + \dot{U} \cdot {}^tU \end{aligned} \tag{7}$$

where $U = U(t) \in SO(N)$ and ${}^tU$ stands for the transpose of U. This is easily verified, since

$$(\mathscr{D}\mathbf{x})' = \dot{\mathbf{x}}' - Y' \cdot \mathbf{x}' = U \cdot (\mathscr{D}\mathbf{x})$$

and the scalar product is invariant under rotations.

Thus, we are describing the particle motion independently of the state of rotation of the coordinate system. It may be related to a fixed reference frame by any matrix $U = U(t)$. To obtain the equations of motion, we first define the momentum vector conjugate to $\mathbf{x}$,

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \dot{\mathbf{x}} - Y \cdot \mathbf{x} \tag{8}$$

Then, we have

$$\dot{\mathbf{p}} = Y \cdot \mathbf{p} - \frac{\partial V}{\partial \mathbf{x}} \tag{9}$$

$$0 = \mathbf{p} \cdot (T^a \cdot \mathbf{x}); \qquad a = 1, \ldots, \binom{N}{2} \tag{10}$$

The momentum conjugate to the y_a is zero, since their time derivatives do not appear in L.

We have the freedom to choose a gauge to analyze the dynamics described by these equations of motion. This is equivalent to choosing a system of coordinates. Especially important to the end of quantization will be the frame at rest with respect to the space, which, in this formalism, is selected by the gauge condition $Y = 0$. First, we have to show that it is possible to impose this condition on the system.

Lemma 1. There exists a gauge transformation $U(t)$ such that, for a given trajectory $(\mathbf{x}(t), Y(t))$ in configuration space, we have

$$\mathbf{x}'(t) = U(t) \cdot \mathbf{x}(t) \tag{11}$$

$$Y'(t) = 0 \tag{12}$$

Proof. From equation (7) we see that given the antisymmetric matrix $Y(t)$, we only have to find a matrix $U(t) \in SO(N)$ satisfying

$$\dot{U} = -U \cdot Y \tag{13}$$

for the condition $Y' = 0$ to be satisfied. In analogy with the case in field theory (Lee, 1981), U is simply expressed as

$$U = \mathbb{T} \exp\left[-\int_0^t dt\, Y(t)\right]$$

In Appendix A this expression is considered in greater detail.

In this gauge all the y_a are zero. We can also choose a rotating coordinate system in such a way as to cancel some of the coordinates of the particle. In this case, the y_a will not be fixed and the number of independent variables is maintained. This new gauge should be related to the former by a gauge transformation. In the following Lemma we prove the existence of another gauge of interest for quantization:

Lemma 2. Given any trajectory $x(t)$, $Y(t)$ of the system, there always exists a gauge transformation such that

$$\mathbf{x}' = (r, \underbrace{0, \ldots, 0}_{N-1}); \qquad r > 0 \tag{14}$$

$$Y' \quad \text{satisfies} \quad (Y')_{ij} \neq 0 \qquad \text{only if} \quad i = 1 \text{ or } j = 1 \tag{15}$$

Proof. By the previous Lemma, we can suppose that the trajectory $\mathbf{x}(t)$, $Y(t)$ satisfies $Y = 0$. We then define the transformation

$$U(t) = R_{(N-1)}(\phi_1, \ldots, \phi_m) \cdot P(\theta_2, \ldots, \theta_N) \tag{16}$$

where $P(\theta_2, \ldots, \theta_N)$ is a matrix defined in Appendix C, $\theta_2(t), \ldots, \theta_N(t)$ are spherical polar angles introduced in Appendix B, and $\phi_1, \ldots, \phi_m$ are parameters defining an $SO(N-1)$ matrix through the parametrization introduced in Appendix C such that

$$R_{(N-1)}(\phi_1, \ldots, \phi_m) = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \mathbb{r} & \\ 0 & & & \end{array}\right) \tag{17}$$

As we shall see, these parameters $\phi_1, \ldots, \phi_m$ are determined by $\theta_2, \ldots, \theta_N$ uniquely.

By definition, P and $R_{(N-1)}$ belong to $SO(N)$; then $U \in SO(N)$ and is an admissible transformation. Using the transformation law of $\mathbf{x}$ and Appendix C, we find that $\mathbf{x}$ already has the desired form

$$\mathbf{x}' = U\mathbf{x} = (r, 0, \ldots, 0) \tag{18}$$

Since $Y=0$, by Lemma 1 we have

$$Y' = \dot{U}\,{}^tU = \dot{R} \cdot {}^tR + R \cdot \dot{P} \cdot {}^tP \cdot {}^tR \tag{19}$$

So far, only P has been used to reduce $\mathbf{x}'$ to the form (14). Since $R_{(N-1)}$ does not alter this form, we shall choose it to obtain (15). We then consider the antisymmetric matrix P^tP,

$$\dot{P} \cdot {}^tP = \left(\begin{array}{c|c} 0 & {}^t\mathbf{\Omega} \\ \hline -\mathbf{\Omega} & p \end{array}\right) \tag{20}$$

where $\mathbf{\Omega} \in \mathbb{R}^{N-1}$ and $p \in \mathbb{R}^{(N-1)\times(N-1)}$ are defined by this last equation. The matrix p is antisymmetric. According to (17), (19), and (20), we have

$$Y' = \left(\begin{array}{c|c} 0 & {}^t\mathbf{\Omega} \cdot {}^t\mathbb{r} \\ \hline -\mathbb{r} \cdot \mathbf{\Omega} & \mathbb{r} \cdot p \cdot {}^t\mathbb{r} \end{array}\right) + \left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \dot{\mathbb{r}} \cdot {}^t\mathbb{r} & \\ 0 & & & \end{array}\right) \tag{21}$$

and we only have to find r such that

$$\mathbb{r} \cdot p \cdot {}^t\mathbb{r} + \dot{\mathbb{r}} \cdot {}^t\mathbb{r} = 0$$

or

$$\dot{\mathbb{r}} = -\mathbb{r} \cdot p$$

with p an antisymmetric $(N-1)\times(N-1)$ matrix. Again, as in Lemma 2, the existence of this $\mathbb{r}$ is shown in Appendix A.

By construction, $\mathbf{\Omega}$ and p, and then $\mathbb{r}$, are determined by the matrix P whose parameters $\theta_2, \ldots, \theta_N$ are the polar angles of $\mathbf{x}$.

We notice that these considerations are based only on symmetry argument, being purely kinematic in nature. In the next section we shall see that the vector $\mathbf{\Omega}$ must be zero due to the equations of motion. However, $\mathbf{\Omega}=0$ cannot be deduced from the symmetry and then cannot be imposed as a gauge condition.

3. QUANTIZATION OF THE MODEL (1)

We shall carry out the quantization in the Schrödinger picture, in the coordinate representation. We shall follow (Lee, 1981) and start from the gauge $Y=0$.

The equation of motion (8) reads in this gauge

$$\dot{\mathbf{p}}=-\frac{\partial V}{\partial \dot{\mathbf{x}}}; \qquad \mathbf{p}=\dot{\mathbf{x}} \tag{22}$$

and is the Euler–Lagrange equation derived from (5) restricted to $Y=0$,

$$L_1=\tfrac{1}{2}\dot{\mathbf{x}}^2-V(|\mathbf{x}|) \tag{23}$$

Equation (9) must be considered as a supplementary equation (Dirac, 1964; Lee, 1981), i.e., a constraint on the motion of the system

$$0=\mathbf{p}\cdot(T^a\cdot\mathbf{x}); \qquad a=1,\ldots,\binom{N}{2} \tag{24}$$

From this set of equations, $\binom{N-1}{2}$ are identically satisfied due to the fact that, since the group $SO(N-1)$ of rotations around $\mathbf{x}$ leave it invariant, its generators must give zero when applied to $\mathbf{x}$ (Prokhorov, 1982). Thus, we are left with $(N+1)$ independent quantities out of the $2N$ components of $\mathbf{p}$, $\mathbf{x}$. Indeed, the solutions to equation (24) are of the form (Prokhorov, 1982)

$$\mathbf{p}=\lambda\mathbf{x} \tag{25}$$

The Hamiltonian formulation results from (23), (24),

$$H=\tfrac{1}{2}\mathbf{p}^2+V(|\mathbf{x}|) \tag{26}$$

$$G^a=\mathbf{p}\cdot(T^a\cdot\mathbf{x})=0 \tag{27}$$

and is a constrained one. Equation (27) is a momentum-dependent constraint. The Poisson bracket of the constraints G^a is given by

$$[G^a, G^b]_P=f^{abc}G^c \tag{28}$$

where the constants f^{abc} satisfy

$$T^a \cdot T^b - T^b \cdot T^a = f^{abc} \cdot T^c$$

To quantize, we impose on the phase-space variables the commutation conditions

$$[x_j, p_k] = i \cdot \delta_{jk} \tag{29}$$

These are the only nonzero commutators. In the coordinate representation, we then have

$$p_k = \frac{1}{i} \cdot \frac{\partial}{\partial x_k} \tag{30}$$

Since the matrices T^a are antisymmetric, the constraints G^a are directly given by its classical expression, without the need for a specific ordering rule. They satisfy the same algebra as the T^a,

$$[G^a, G^b] = i \cdot f^{abc} \cdot G^c \tag{31}$$

The quantum counterpart of equations (26), (27) is then

$$H = -\frac{1}{2} \frac{\partial^2}{\partial x_k \, \partial x_k} + V(|\mathbf{x}|) \tag{32}$$

$$G^e \psi \equiv (T^e)_{jk} \frac{1}{i} \frac{\partial}{\partial x_j} x_k \psi = 0 \tag{33}$$

where $\psi = \psi(\mathbf{x}, t)$ is the wave function of the system.

We see, then, that in this gauge the quantum version of the model is the same as the one we would have obtained had we started from Lagrangian (1), except for the conditions (33). These last equations restrict the state vector ψ to a subspace of the Hilbert space of square-integrable functions depending on $\mathbf{x}$. They simply state that we must consider only wave functions with zero angular momentum, and are the only trace left of gauge invariance once we have fixed the $Y = 0$ gauge (Lee, 1981). Finally, we should remark that equations (33) are consistent in virtue of the commutation relations (31) (Dirac, 1964).

4. QUANTIZATION OF THE MODEL (2)

In the previous section we have seen the quantization within the $Y = 0$ gauge condition. It is almost identical to the quantization of the nongauge system described by equation (1), the fundamental operators of the theory being obtained by a straightforward application of the usual quantization rules. However, we still have the constraints (33), which complicate things a bit. In more realistic systems, such constraints may constitute an almost insurmountable obstacle.

It is at this point where we can exploit the invariance of the system. Starting from the results of Section 3, we shall quantize our model in the gauge of Lemma 2 (Section 2). It will turn out that this is equivalent to quantizing the model of equation (1) in polar coordinates, in the same way as in Section 3 we arrived at its quantization in Cartesian coordinates. This is to be expected, since, as pointed out in Section 2, choosing the gauge $x_j = 0$ $(j \geq 2)$ means choosing a rotating coordinate system that is synchronized with the particle so as to keep it always on the x axis. Obviously, we can identify the coordinate x_1 with the radial coordinate and the orthogonal axes $x_2, \ldots, x_n$ with the transverse directions on which the polar angles vary.

The Lagrangian (5) restricted by (14), (15) reads

$$L_2 = \tfrac{1}{2}\dot{r} - \tfrac{1}{2}(Y_{1i}Y_{i1})r^2 - V(r) \tag{34}$$

The index i runs from 2 to N, since $Y_{11} = 0$. The Y_{i1} are the components of $\mathbf{\Omega}$ [equation (20)]. Their time derivatives do not enter the Lagrangian L_2, so we have

$$\frac{\partial L_2}{\partial Y_{i1}} = r^2 Y_{i1} = 0 \tag{35}$$

which was anticipated in the Remark in the end of Section 2.

Therefore, the equation of motion for r is derived from

$$L_2' = \tfrac{1}{2}\dot{r}^2 - V(r) \tag{36}$$

Our quantum state space is now that of functions $\psi = \psi(r)$ depending on $r \geq 0$. The quantum Hamiltonian is obtained from (32) by using the transformations law (10) with U given by (16). Since the parameters $\theta_2, \ldots, \theta_N$ are the spherical angles $\mathbf{x}$, we have

$$\mathbf{x} = r\hat{e}(\theta_2, \ldots, \theta_N) \tag{37}$$

with $\hat{e}(\theta_2, \ldots, \theta_N)$ given in Appendix B. We need not worry about Y, since we know that it is dynamically zero. By applying the chain rule to (32) with (37), we obtain

$$H_2 = -\frac{1}{2}\left[\frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r}\right] + \frac{1}{r^2}\hat{L}_\theta + V(r) \tag{38}$$

where $\hat{L}_\theta$ is an operator depending only on $\theta_2, \ldots, \theta_N$. For functions ψ independent of the angular variables, H_2 reduces to

$$H_2 = -\frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r}\right) + V(r) \tag{39}$$

The inner product in the state space is given also by transformation (37). In Appendix D it is shown that

$$(\phi, \psi) = \int dr\, r^{N-1} \phi^* \psi \tag{40}$$

up to a constant that is the surface of an $(N-1)$-dimensional unit sphere. The explicit form of $\hat{L}_\theta$ is also shown there.

In this gauge the Hamiltonian has the simpler form (39), and the constraints are easily incorporated into the wave function by eliminating its angular dependence. H_2 is easily recognized as the radial part of the Laplacian in spherical coordinates. Similarly, by imposing inadequate gauge condition, other curvilinear coordinates may be reached.

5. FINAL REMARKS

We have presented here a simple model and its quantization as a worked example of a gauge-invariant system. The traditional Lagrangian point of view used, where the gauge is fixed before passing to the Hamiltonian, is most simple and economical and may be applied to a wide class of problems, including the theory of Yang–Mills fields for which it was originally formulated (Lee, 1981). We hope that its salient features have been clearly illustrated.

This generalization to an arbitrary number of spatial dimensions may also be the starting point of a $1/N$ perturbative expansion. Such a development around $N = \infty$ poses interesting questions as to its effects on the gauge group and algebra acting on the system which are currently being investigated.

The method breaks down when the gauge transformations do not possess a group structure, i.e., when there are structure functions rather than simply constants. Phase-space methods of broader scope must be used there, which may also be exemplified by this kind of mechanical model (Alessandrini, 1989).

APPENDIX A

In this Appendix we show that the equation

$$\dot{A} = -A \cdot Y(t) \tag{A.1}$$

where A and Y are $(k \times k)$ matrices and $Y(t)$ is antisymmetric, possesses a solution $A(t)$ in $SO(K)$.

To this end, we consider the matrix

$$A(t) = \mathbb{T}\left[\exp - \int_0^t dt\ Y(t)\right]$$

where $\mathbb{T}$ is the chronological ordering operator defined as follows.

Consider the function

$$I(t_1, \ldots, t_n) = \begin{cases} 1 & \text{if} \quad t_1 < t_2 < \cdots < t_n \\ 0 & \text{otherwise} \end{cases} \tag{A.2}$$

Then we define

$$\mathbb{T}[Y(t_1) \cdots Y(t_n)] = \sum_{\sigma_n} I(t_{i_1}, \ldots, t_{i_n}) Y(t_{i_1}) \cdots Y(t_{i_n})$$

where the sum extends over all possible permutations σ_n applying $(1, 2, \ldots, n)$ on $(i_1, i_2, \ldots, i_n)$.

The matrix A is then given by the series

$$A(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^t dt_1 \cdots \int_0^t dt_n\, \mathbb{T}[Y(t_1) \cdots Y(t_n)]$$

or, using the definition of $\mathbb{T}$ and changing the order of integration,

$$A(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \times \sum_{\sigma_n} \int_0^t dt_{i1} \cdots \int_0^t dt_{in}\, I(t_{i_1}, \ldots, t_{i_n}) Y(t_{i_1}) \cdots Y(t_{i_n})$$

From the definition of $I(t_1, \ldots, t_n)$ it follows that

$$A(t) = \sum_{n=0}^{\infty} (-1)^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_3} dt_2 \int_0^{t_2} dt_1\ Y(t_1) \cdots Y(t_n)$$

Hence

$$\dot{A}(t) = \sum_{n=1}^{\infty} (-1)^n \int_0^t dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \cdots \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \cdot Y(t_1) \cdots Y(t_{n-1}) \cdot Y(t)$$

which may be rewritten in the form

$$\dot{A}(t) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!}$$
$$\times \sum_{\sigma_{n-1}} \int_0^t dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \cdots \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \; Y(t_1) \cdots Y(t_{n-1}) \cdot Y(t)$$

to yield

$$\dot{A}(t) = -A(t) \cdot Y(t)$$

It remains to be shown that $A(t) \in SO(k)$. It is easily seen that

$${}^tA = \mathbb{T} \exp \int_0^t dt \; Y(t)$$

where $\mathbb{T}$ is the anti-chronological ordering operator defined by reversing the inequalities in equation (A.2). It then follows that

$$A(t)\,{}^tA(t) = A(0)\,{}^tA(0) = 1$$

APPENDIX B. SPHERICAL COORDINATES IN $\mathbb{R}^N$

The spherical polar angles used in the text are defined by the following parametrization of the $(N-1)$-dimensional sphere $S^{N-1} \subset \mathbb{R}^n$:

$$e: \quad [0, 2\pi) \times [-\pi/2, \pi/2]^{N-2} \to S^{N-1}$$
$$e(\theta_2, \ldots, \theta_N) = (y_1, \ldots, y_N)$$

with

$$\begin{aligned}
y_1 &= \cos\theta_N \cdot \cos\theta_{N-1} \cdot \cdots \cdot \cos\theta_3 \cdot \cos\theta_2 \\
y_2 &= \cos\theta_N \cdot \cos\theta_{N-1} \cdot \cdots \cdot \cos\theta_3 \sin\theta_2 \\
y_3 &= \cos\theta_N \cdot \cos\theta_{N-1} \cdot \cdots \cdot \cos\theta_4 \cdot \sin\theta_3 \\
&\vdots \\
y_{N-2} &= \cos\theta_N \cdot \cos\theta_{N-1} \cdot \sin\theta_{N-2} \\
y_{N-1} &= \cos\theta_N \cdot \sin\theta_{N-1} \\
y_N &= \sin\theta_N
\end{aligned}$$

The vector $\mathbf{x} \in \mathbb{R}^N$ is then expressed as

$$\mathbf{x} = r \cdot e(\theta_2, \ldots, \theta_N); \qquad r = (\mathbf{x} \cdot \mathbf{x})^{1/2} \geq 0$$

APPENDIX C

We define the matrix P as

$$P(\theta_2, \ldots, \theta_N) = B_{1N}(\theta_N) \cdot B_{1(N-1)}(\theta_{N-1}) \cdot \cdots \cdot B_{12}(\theta_2)$$

where the matrices B_{1k} are given by $(p > k)$

$$(B_{pk}(\theta_k))_{ij} = \begin{cases} 1 & \text{if } \; k \neq i = j \neq p \\ \cos\theta_k & \text{if } \; i = j = p \text{ or } i = j = k \\ \sin\theta_k & \text{if } \; i = p \text{ and } j = k;\, i \neq j \\ -\sin\theta_k & \text{if } \; i = k \text{ and } j = p;\, i \neq j \\ 0 & \text{otherwise} \end{cases}$$

$k = p, \ldots, N$. It is immediately verified that

$$P(\theta_2, \ldots, \theta_k) \cdot e(\theta_2, \ldots, \theta_k) = (1, 0, \ldots, 0)$$

The most general matrix $R_{(N)} \in SO(N)$ is of the form

$$R_{(N)} = R^*_{(N-1)} \cdot P(\theta_2, \ldots, \theta_N)$$

with $[R_{(N-1)} \in SO(N-1)]$

$$R^*_{(N-1)} = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & R_{(N-1)} & \\ 0 & & & \end{array}\right)$$

This definition is iterative,

$$R_{(N-1)} = R^*_{(N-2)} \cdot P'(\theta_2, \ldots, \theta_{N-1}); \quad \text{etc.}$$

A basis for the algebra is obtained by differentiating $R_{(N)}$ with respect to each parameter and evaluating at the identity ($\theta = 0$, for all θ). This basis is

$$(A^{ij})_{Pq} = \delta_{iP}\delta_{jq} - \delta_{iq}\delta_{jP} \; (i < j)$$

Therefore

$$\mathrm{tr}(A^{ij} \cdot A^{lm}) = -2\delta_{il}\delta_{jm}$$

By construction we have $\binom{N}{2}$ parameters θ, out of which $(N-1)$ are longitude angles ($\theta \in [0, 2\pi)$) and $\binom{N-1}{2}$ are latitude angles

$$(\theta \in [-\pi/2, \pi/2])$$

Further details can be found in group theory textbooks (Murnaghan, 1938).

APPENDIX D

We consider the transformation

$$\mathbf{x} \to \mathbf{x}' = (x_1', 0, \ldots, 0); \qquad x_1' = r \geq 0$$

This transformation is performed by means of the matrix $P(\theta_2 \cdots \theta_N)$ defined in Appendix C,

$$x_k' = (P)_{kj} \cdot x_j = r\delta_{k1}; \qquad r = (\mathbf{x} \cdot \mathbf{x})^{1/2} \geq 0$$

Since P is orthogonal,

$$x_k = (P)_{jk} \cdot x_j' = (P)_{1k} \cdot r$$

Then

$$\frac{\partial x_k}{\partial r} = (P)_{1k}$$

$$\frac{\partial x_k}{\partial \theta_j} = r\frac{\partial (P)_{1k}}{\partial \theta_j}; \qquad j = 2, \ldots, N$$

Calling V the matrix whose elements are these derivatives, it can be proven by induction that

$$\det V = r^{N-1}(\cos\theta_3)^{N-2} \cdot (\cos\theta_4)^{N-3} \cdot \cdots \cdot \cos\theta_N$$

This gives the integration measure (40) after integration over the angular variables. The metric tensor in configuration space $\mathbb{R}^N$ in these coordinates is

$$g_{ij} = \frac{\partial x_k}{\partial q_i} \cdot \frac{\partial x_h}{\partial q_j}; \qquad q_1 = r; \quad q_j = \partial_j \quad \text{for} \quad 2 \leq j \leq N$$

Explicitly, we have

$$g_{11} = 1$$

$$g_{(kk)} = r^2(\cos^2\theta_N \cdot \cdots \cdot \cos^2\theta_{k+1}), \qquad 2 \leq k < N$$

$$g_{NN} = r_2$$

(bracketed indices are not to be summed over)

$$g_{ij} = 0 \qquad \text{for} \quad i \neq j$$

Writing

$$g^{ij}=(g^{-1})_{ij}$$

$$g=\det g_{ij}$$

We obtain the well-known expression

$$\nabla^2=\frac{1}{\sqrt{g}}\frac{\partial}{\partial q_i}\left(\sqrt{g}\,g^{ij}\frac{\partial}{\partial q_j}\right)$$

Using the expression for g_{ij}, we obtain

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial r}\sqrt{g}\,g^{11}\frac{\partial}{\partial r}=\frac{1}{r^{N-1}}\frac{\partial}{\partial r}r^{N-1}\frac{\partial}{\partial r}=\frac{\partial^2}{\partial r^2}+\frac{N-1}{r}\frac{\partial}{\partial r}$$

$$\begin{aligned}&\frac{1}{\sqrt{g}}\frac{\partial}{\partial\theta_{(k)}}\sqrt{g}\,g^{(kk)}\frac{\partial}{\partial\theta_{(k)}}\\&\quad=\frac{1}{r^2}\frac{1}{\cos^2\theta_n\cdots\cos^2\theta_{k+1}}\\&\qquad\times\frac{1}{(\cos\theta_{(k)})^{N-k+1}}[\partial/\partial\theta_{(k)}](\cos\theta_{(k)})^{N-k+1}\,\partial/\partial\theta_{(k)}\\&\qquad(2<k<N)\end{aligned}$$

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial\theta_2}\sqrt{g}\,g^{22}\frac{\partial}{\partial\theta_2}=\frac{1}{r^2}\frac{1}{\cos^2\theta_n\cdot\,\cdots\,\cdot\cos^2\theta_3}\frac{\partial^2}{\partial\theta_2^2}$$

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial\theta_N}\sqrt{g}\,g^{NN}\frac{\partial}{\partial\theta_N}=\frac{1}{r^2\cos\theta_N}\frac{\partial}{\partial\theta_N}\cos\theta_N\frac{\partial}{\partial\theta_N}$$

This gives an expression for the operator $\hat{L}_\theta$. Alternatively, it may be obtained from (32) by using the chain rule.

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